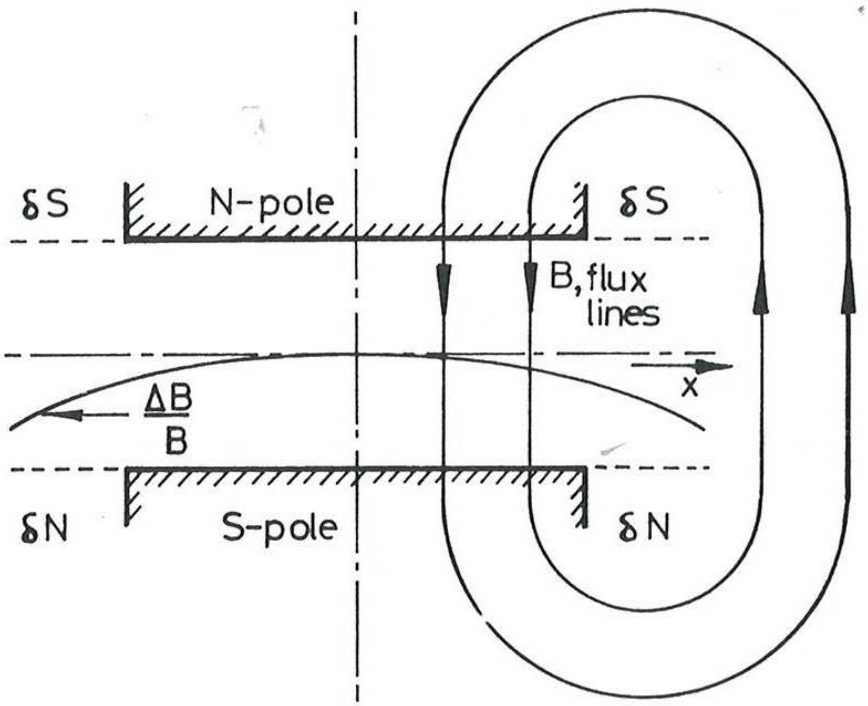
Nonlinear Beam Dynamics I

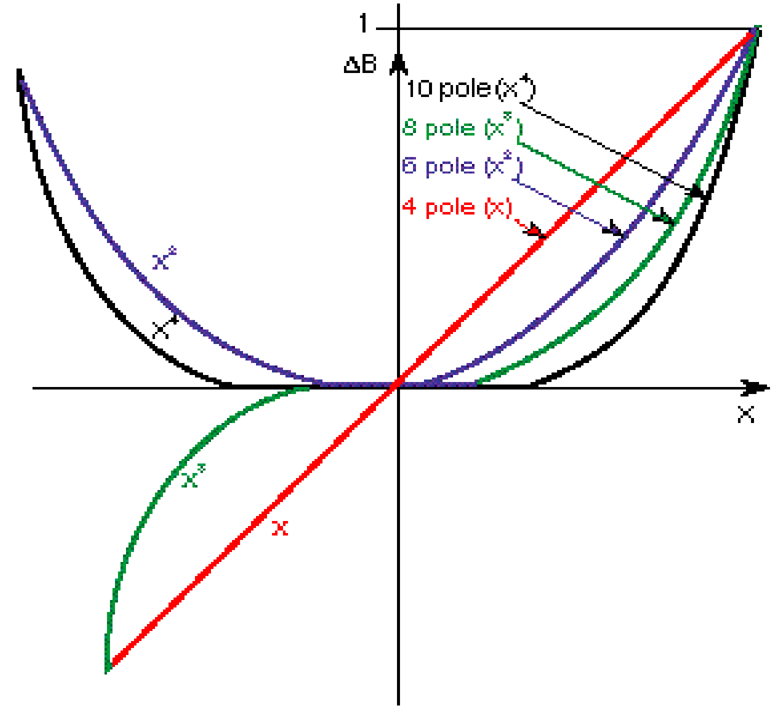
- Nonlinear magnetic multipoles
- Source of chromaticity
- Chromaticity correction with sextupoles
- Phenomenology of nonlinear motion
- Simplified treatment of resonances (stopband concept)

Nonlinear Beam Dynamics II

- Hamiltonian with sextupoles and resonance driving terms
- Tracking, Dynamic Aperture and Frequency Map Analysis

Multipolar expansion of magnetic field





The on axis magnetic field can be expanded into multipolar components (dipole, quadrupole, sextupole, octupoles and higher orders)

$$B_y = B_0 + \frac{1}{1!}\frac{\partial B_y}{\partial x}x + \frac{1}{2!}\frac{\partial^2 B_y}{\partial x^2}x^2 + \frac{1}{3!}\frac{\partial^3 B_y}{\partial x^3}x^3 + \ldots$$

Hill's equation with nonlinear terms

Including higher order terms in the expansion of the magnetic field

$$B_y + iB_x = B_0\rho_0\left[\sum_{n=1}^{M}\frac{k_n(s)+ij_n(s)}{n!}(x+iy)^n\right]$$

$$k_n = \frac{1}{B_0\rho_0}\left.\frac{\partial^n B_y}{\partial x^n}\right|_{(0,0)}$$ normal multipoles

$$j_n = \frac{1}{B_0\rho_0}\left.\frac{\partial^n B_x}{\partial x^n}\right|_{(0,0)}$$ skew multipoles

the Hill's equations acquire additional nonlinear terms

$$\frac{d^2x}{ds^2}+\left(\frac{1}{\rho^2(s)}-k_1(s)\right)x = \mathrm{Re}\left[\sum_{n=2}^{M}\frac{k_n(s)+ij_n(s)}{n!}(x+iy)^n\right]$$

$$\frac{d^2y}{ds^2}+k_1(s)y = -\mathrm{Im}\left[\sum_{n=2}^{M}\frac{k_n(s)+ij_n(s)}{n!}(x+iy)^n\right]$$

No analytical solution available in general:

the equations have to be solved by tracking or analysed perturbatively

Quadrupoles: chromatic aberrations

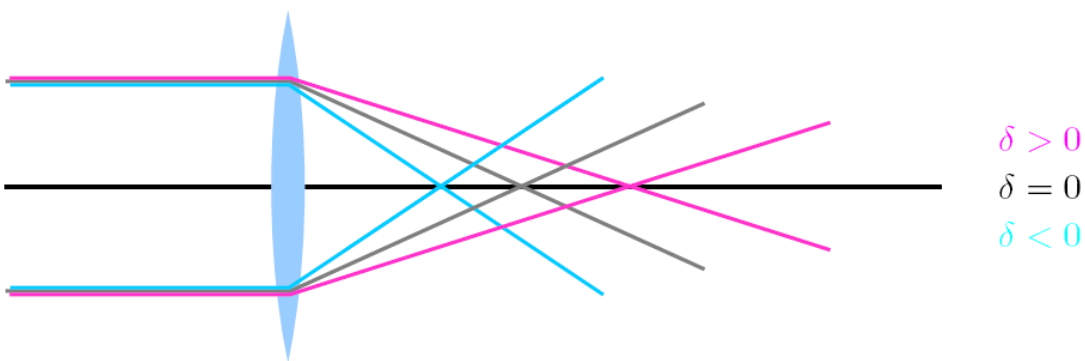
Length L

$$(B\rho)[Tm] = \frac{p}{e} = 3.3356\ Tm \cdot E[GeV]$$

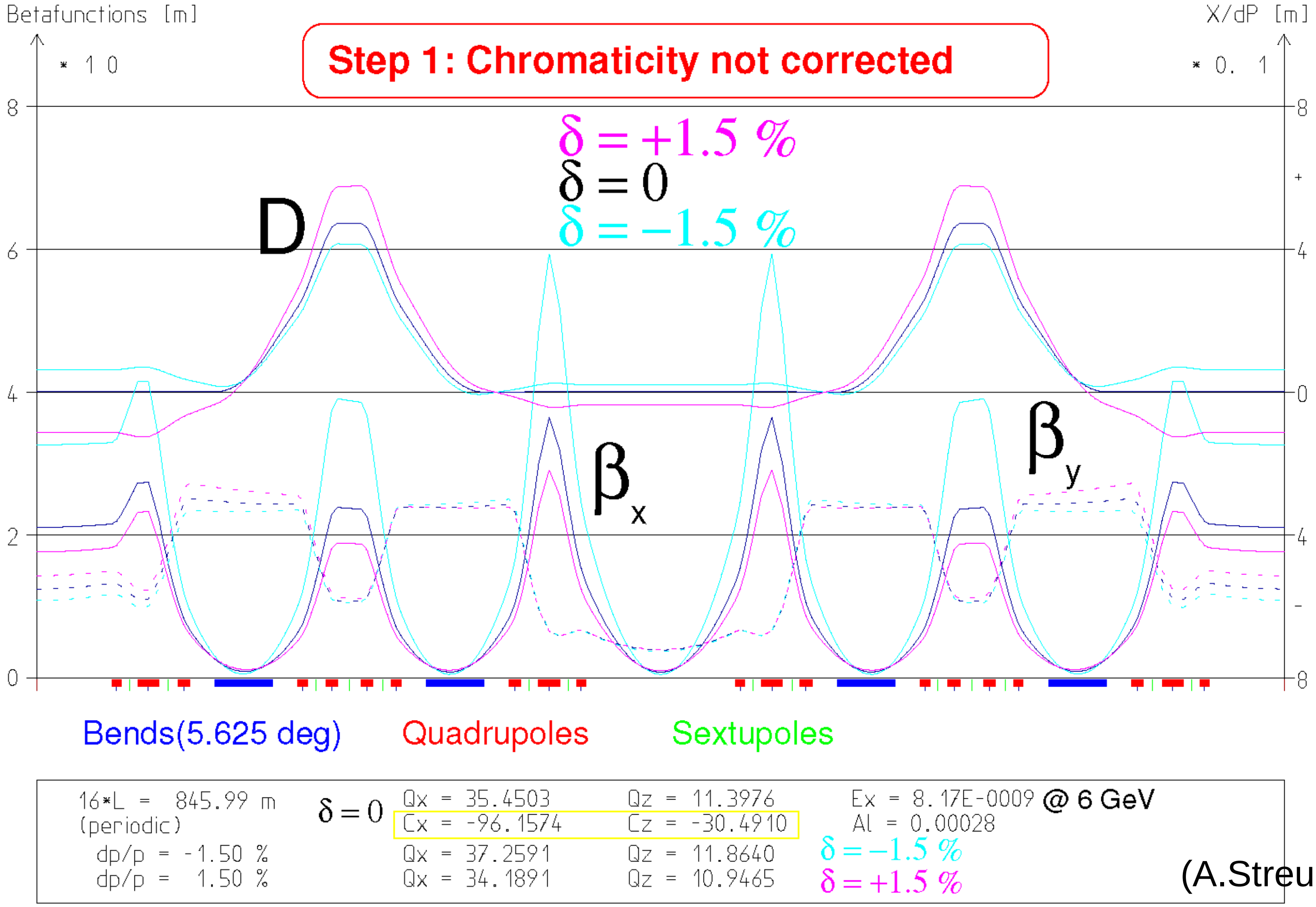
Strength $b_2 = \frac{1}{(B\rho)} \frac{\partial B_y}{\partial x}$

Kicks on particle: $\Delta x' = -b_2 L x$ $\Delta y' = b_2 L y$

Chromatic aberration: $b_2(\delta) = \frac{b_2}{(1+\delta)} \approx b_2(1-\delta)$ $\delta = \frac{\Delta p}{p}$



Example: test lattice



(A.Streun, PSI)

ESRF lattice with original optics for dispersion free straights

Chromaticity

(gradient error $\Delta b_2\, ds$) x (one turn matrix M) = (new one turn matrix $\underline{M}$)

Gradient error due to chromatic aberration: $\pm\,\Delta b_2 = \mp\, b_2\delta$ (hor./vert.)

$$\begin{pmatrix} 1 & 0 \\ \pm b_2\delta\, ds & 1 \end{pmatrix} \times \begin{pmatrix} \cos 2\pi Q & \beta \sin 2\pi Q \\ -\frac{1}{\beta}\sin 2\pi Q & \cos 2\pi Q \end{pmatrix} = \begin{pmatrix} \cos 2\pi \underline{Q} & \beta \sin 2\pi \underline{Q} \\ -\frac{1}{\beta}\sin 2\pi \underline{Q} & \cos 2\pi \underline{Q} \end{pmatrix}$$

$$\tfrac{1}{2} Tr(\underline{M}) = \cos 2\pi \underline{Q} = \cos 2\pi (Q + \Delta Q) = \cos 2\pi Q \pm \tfrac{1}{2} b_2 \delta \beta \sin 2\pi Q\, ds$$

$$\Delta Q << 1 \quad \rightarrow \quad \Delta Q = \mp \tfrac{1}{4\pi} b_2 \delta\, \beta\, ds$$

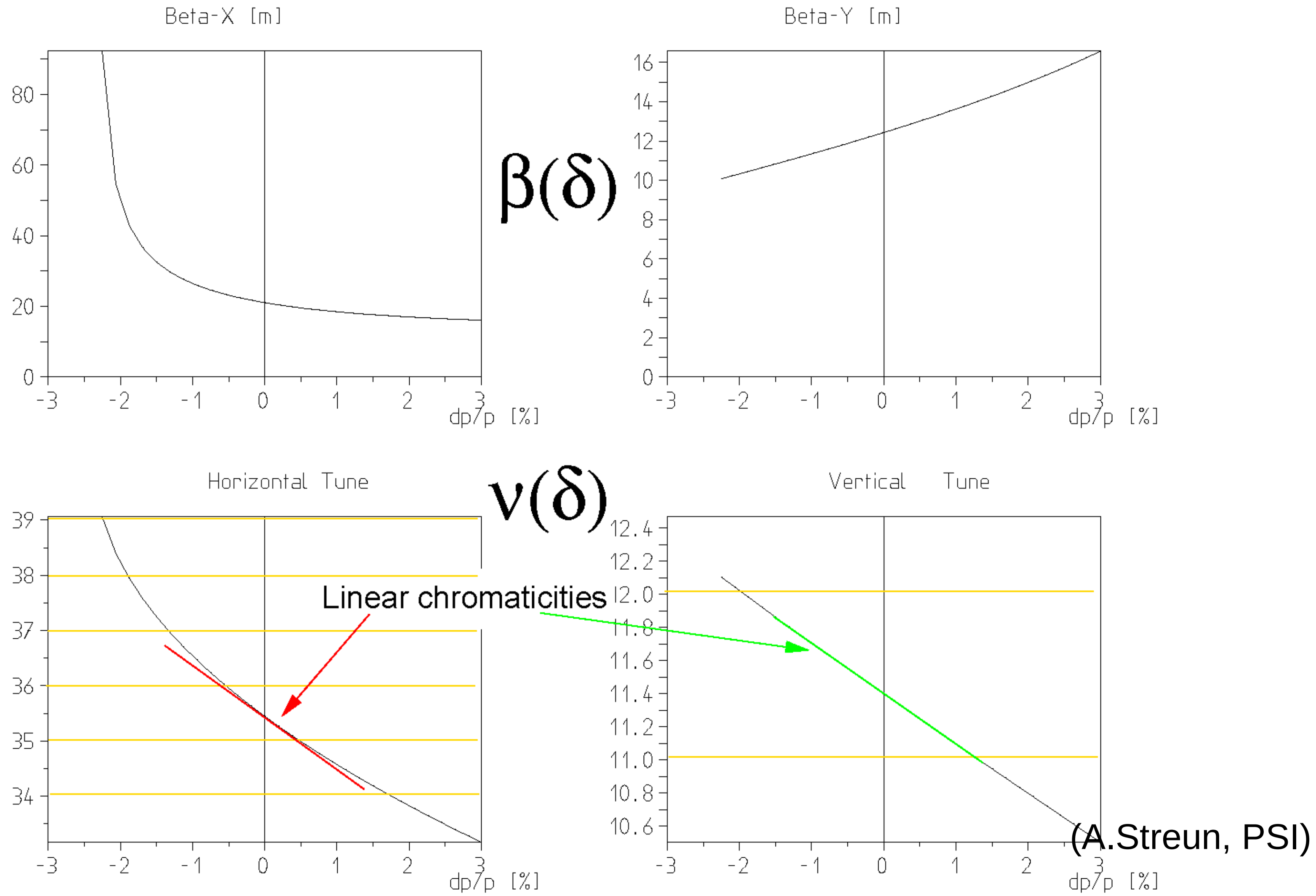
Natural chromaticity: $$\xi_{x,y} = \frac{\Delta Q}{\delta} = \mp \tfrac{1}{4\pi} \oint b_2(s)\, \beta((s)\; ds$$

Light source: Small emittance $\rightarrow$ Strong quadrupoles $\rightarrow$ Large and negative natural chromaticity ($\xi_x \approx$ -50 … -100)

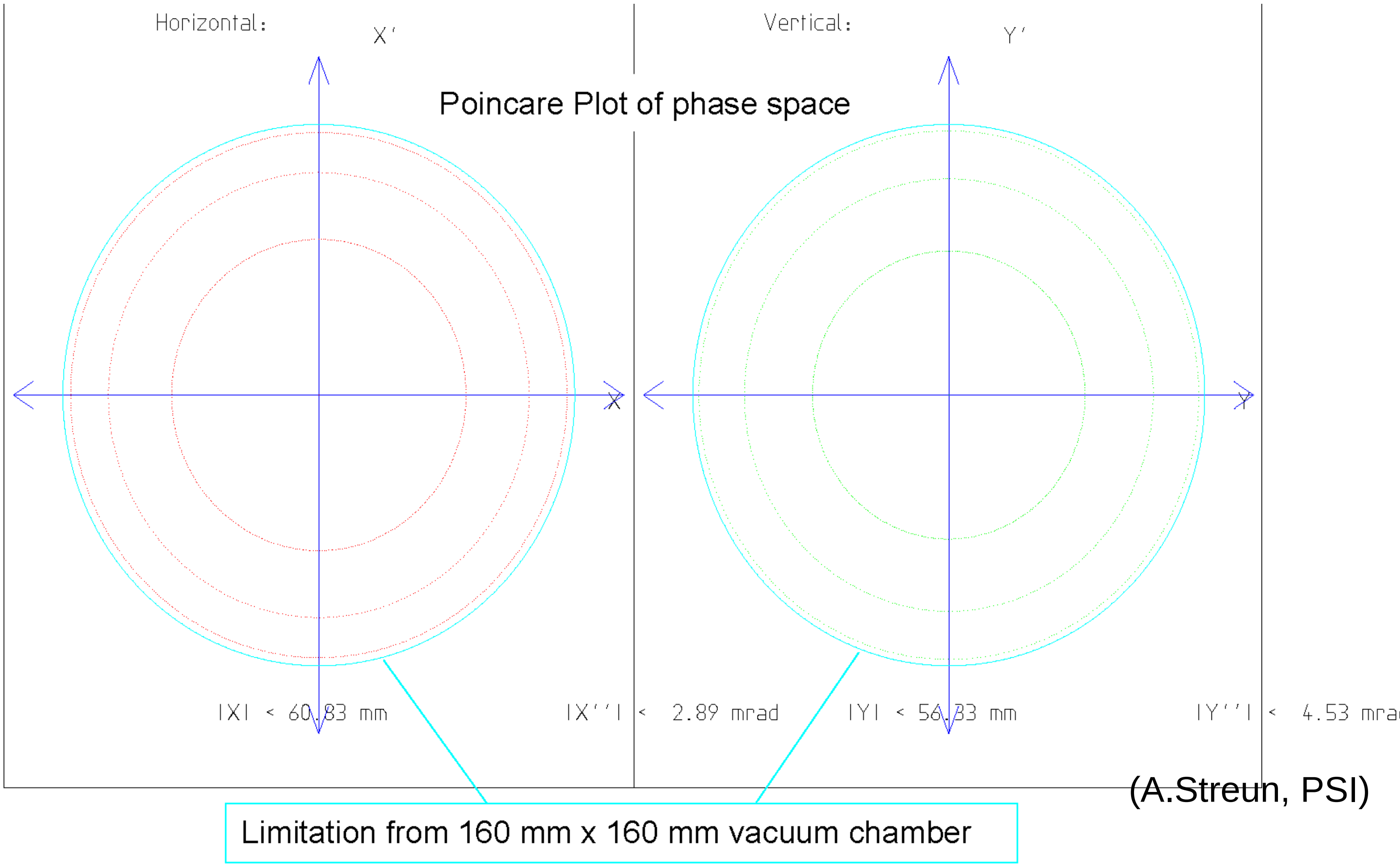
→ Head-tail instability

→ low energy acceptance

Example: natural chromaticity (I)



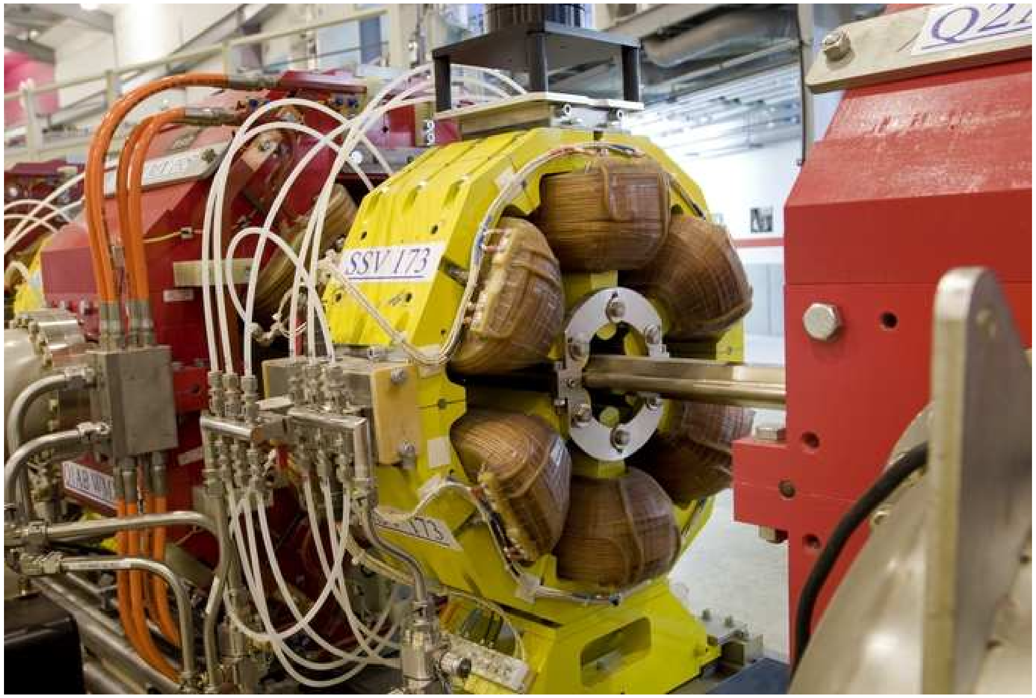
Example: natural chromaticity (II)

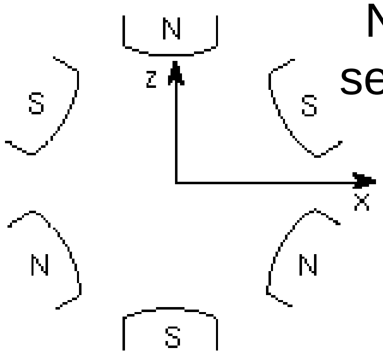


(A.Streun, PSI)

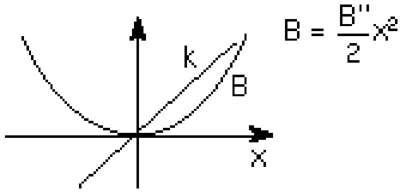
Sextupole magnets

Nonlinear magnetic fields are introduced in the lattice (chromatic sextupoles)



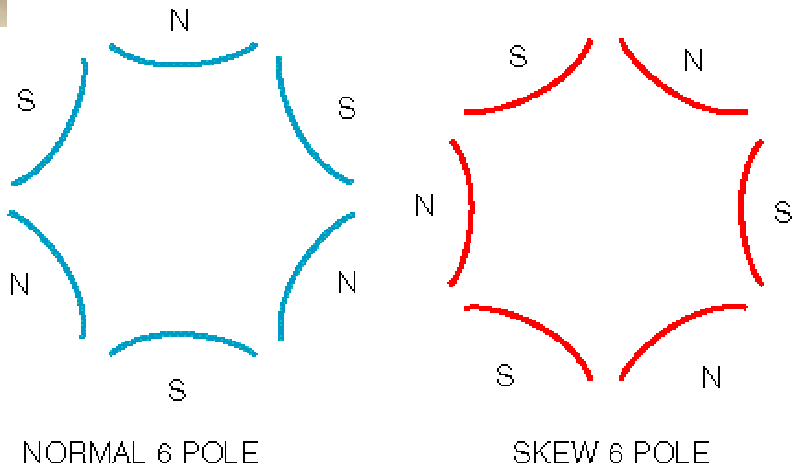


Normal sextupole



$$B_y = \frac{1}{2}\frac{\partial^2 B_y}{\partial x^2}(x^2 - y^2)$$ Normal sextupole

$$B_y = \frac{1}{2}\frac{\partial^2 B_x}{\partial x^2}\cdot 2xy$$ Skew sextupole

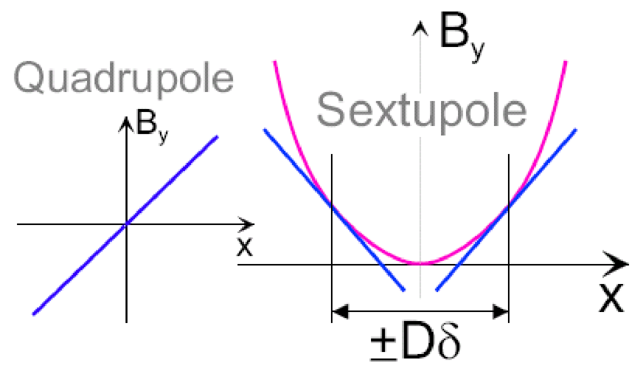


Chromaticity correction with sextupoles

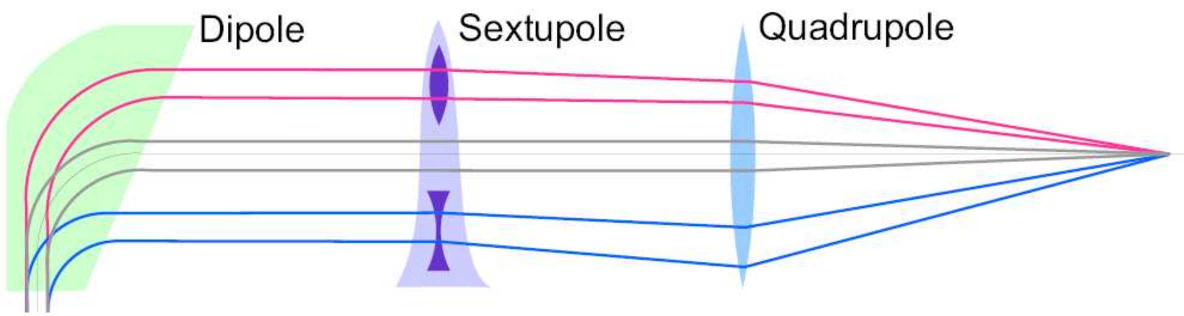
Chromaticity correction

Sextupole: $B_y(x) = \frac{1}{2} B'' x^2$

Local gradient: $B'_y(x) = B'' x$



Strong “chromatic” sextupoles guarantee the focussing of off-energy particles



strong sextupoles have a significant impact on the electron dynamics

$\rightarrow$ additional “harmonic” sextupoles are required to correct nonlinear perturbations

Chromaticity correction

Quadrupole:

$$b_2 = \frac{1}{(B\rho)}\frac{\partial B_y}{\partial x}$$

$$\Delta x' = -b_2 L x$$

$$\Delta y' = b_2 L y$$

Sextupole:

$$b_3 = \frac{1}{2}\frac{1}{(B\rho)}\frac{\partial^2 B_y}{\partial x^2}$$

$$\Delta x' = -b_3 L (x^2 - y^2)$$

$$\Delta y' = 2 b_3 L x y$$

Chromatic aberrations: $b_n(\delta) = \dfrac{b_n}{(1+\delta)} \approx b_n(1-\delta)$

Sextupole in dispersive regions: $x \to x + D\delta \qquad y \to y$

Kicks on a particle (keep up to second order in products of x, y, δ):

$$\Delta x' = -b_2 L x + [b_2 L]\delta x$$

$$\Delta y' = +b_2 L y - [b_2 L]\delta y$$

$$\Delta x' = -[b_3 L D]\delta x - b_3 L (x^2 - y^2) - b_3 L D^2 \delta^2$$

$$\Delta y' = +[b_3 L D]\delta y + 2 b_3 L x y$$

→ Chromaticity correction for $2b_3LD = b_2L$ (good)

→ non linear kicks… (bad)

Chromaticity correction

$$\xi_{x,y} = \pm \tfrac{1}{4\pi} \oint [2b_3(s)D(s) - b_2(s)]\beta_{x,y}(s)ds$$

$$= \pm \tfrac{1}{4\pi} \left(\sum_{sext} 2b_3 L \beta_{x,y} D - \sum_{quads} b_2 L \beta_{x,y}\right) = 0 \quad \text{(or positive)}$$

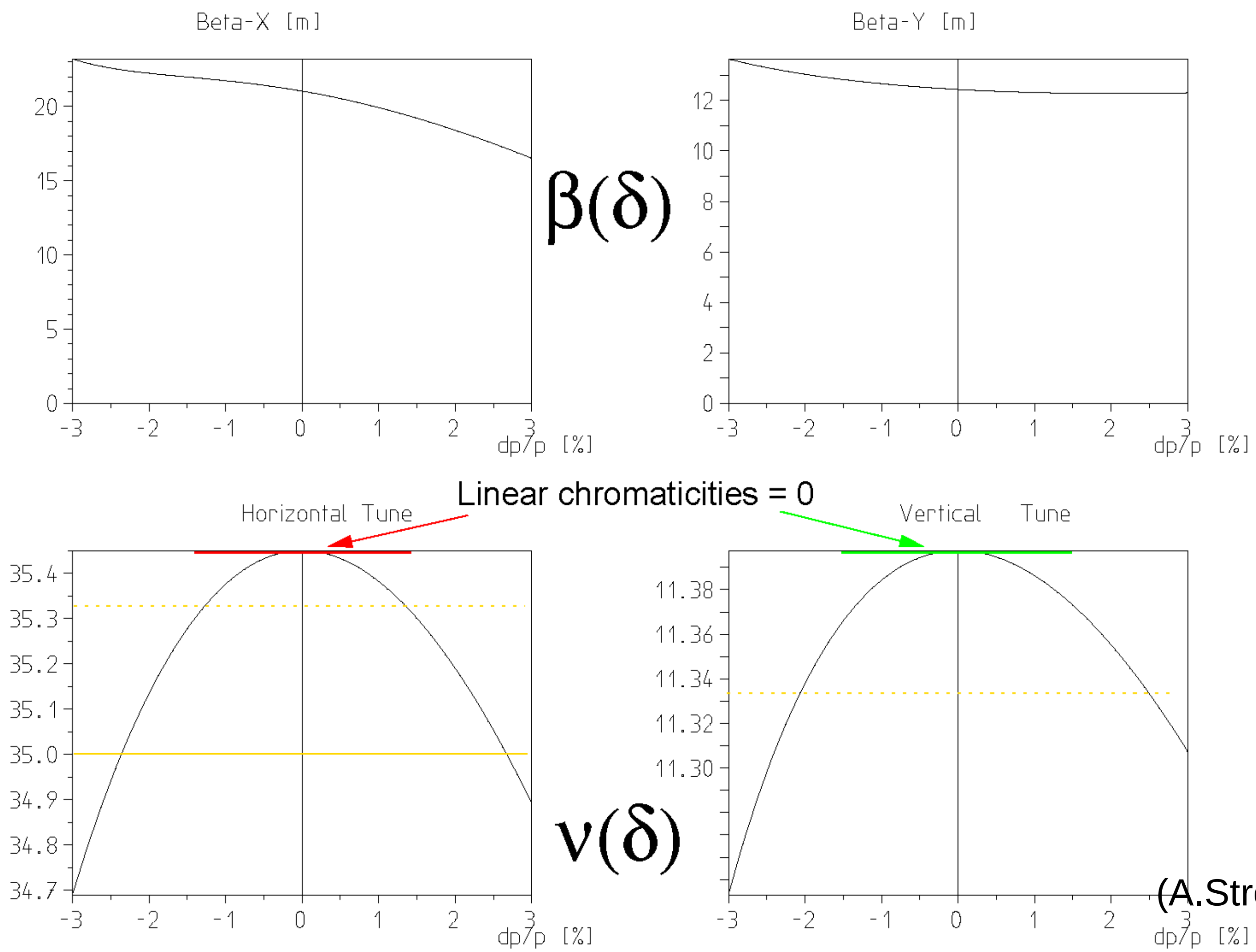
Linear system: 2 families of sextupoles SF, SD

$$\tfrac{1}{2\pi}\begin{pmatrix} +\sum_{SF} \beta_x D & +\sum_{SD} \beta_x D \\ -\sum_{SF} \beta_y D & -\sum_{SD} \beta_y D \end{pmatrix}_{2\times 2} \times \begin{pmatrix} (b_3 L)_{SF} \\ (b_3 L)_{SD} \end{pmatrix}_{1\times 2} = \tfrac{1}{4\pi}\begin{pmatrix} +\sum_{quads} (b_2 L)\beta_x \\ -\sum_{quads} (b_2 L)\beta_y \end{pmatrix} \Rightarrow \quad (b_3 L)_{SF},\ (b_3 L)_{SD}$$

→ SF locations: $D\uparrow, \beta_x\uparrow, \beta_y\downarrow$

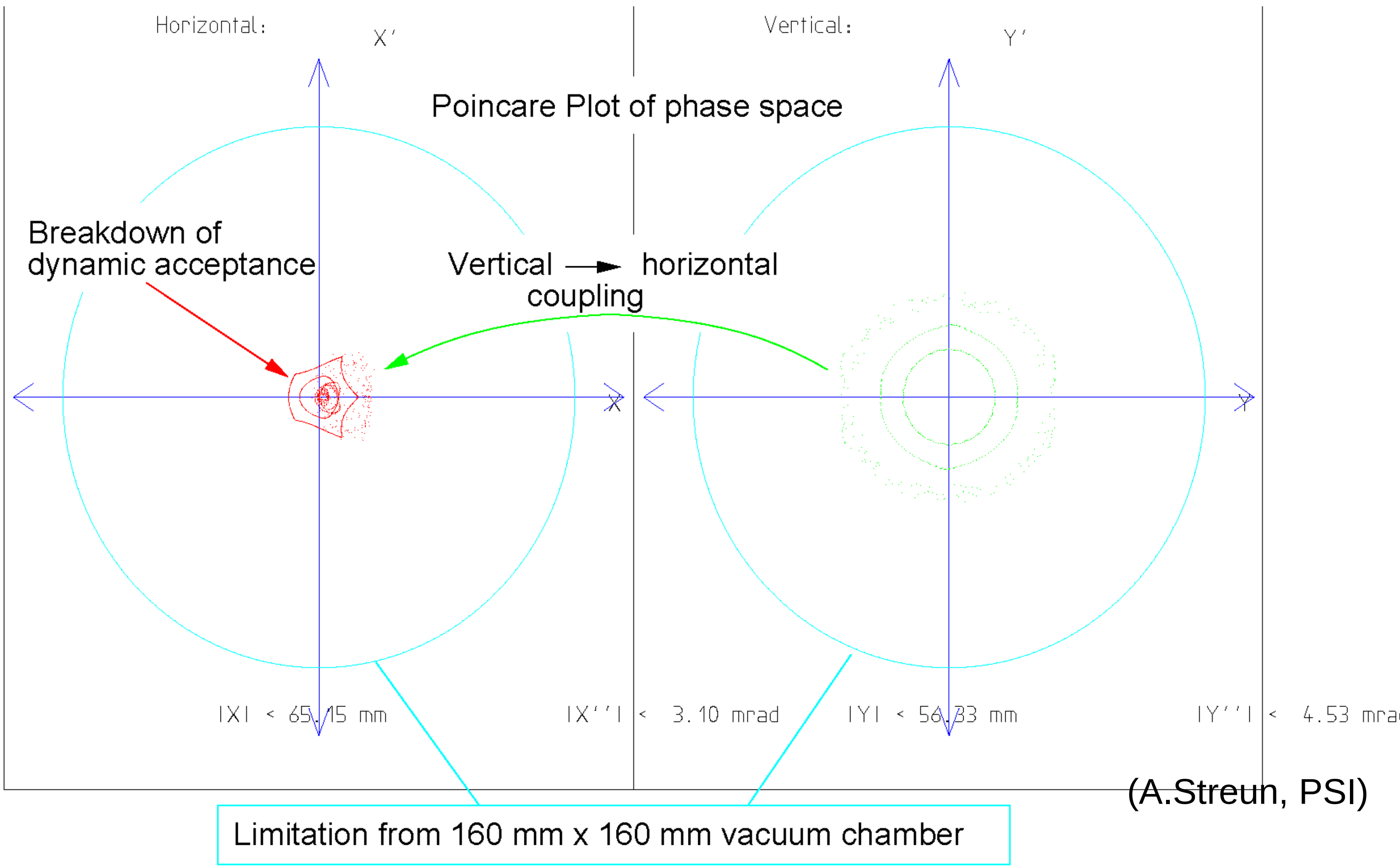
→ SD locations: $D\uparrow, \beta_x\downarrow, \beta_y\uparrow$

Example: chromaticity correction (I)



(A.Streun, PSI)

Example: chromaticity correction (II)

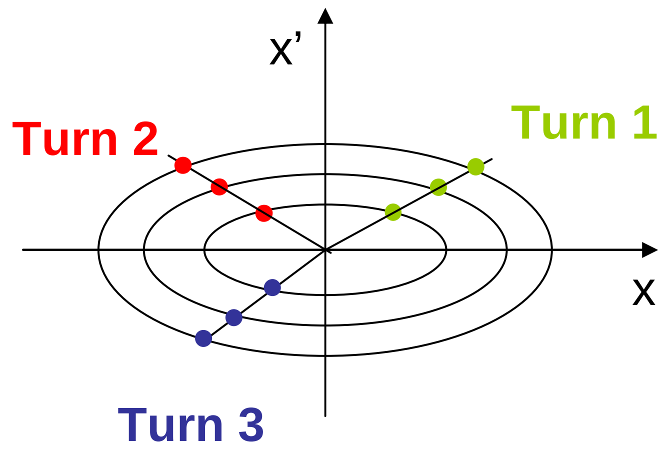


(A.Streun, PSI)

Phenomenology of nonlinear motion (I)

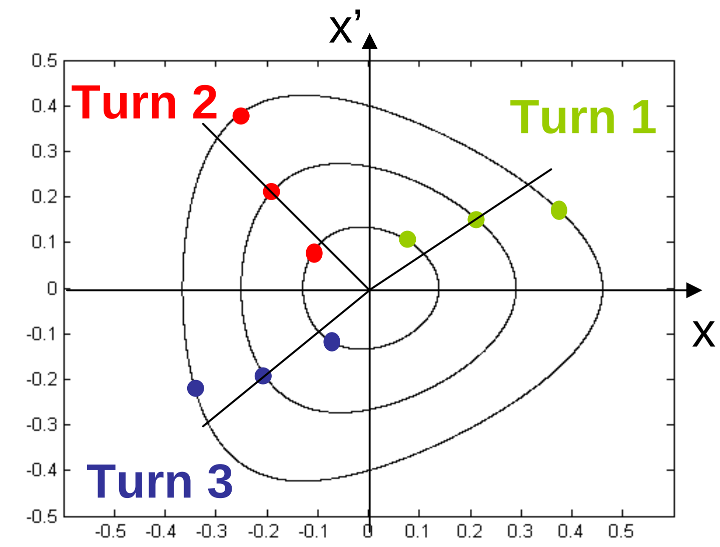
The orbit in phase space for a system of **linear Hill's equation** are ellipses (or circles)

The frequency of revolution of the particles is the same on all ellipses



The orbit in the phase space for a system of **nonlinear Hill's equations** are no longer simple ellipses (or circles);

The frequency of oscillations depends on the amplitude



Resonances

When the betatron tunes satisfy a **resonance relation**

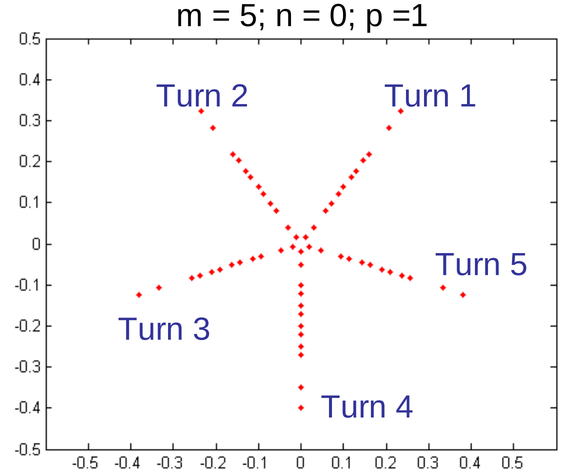
$$mQ_x + nQ_y = r$$

the motion of the charged particle repeats itself periodically

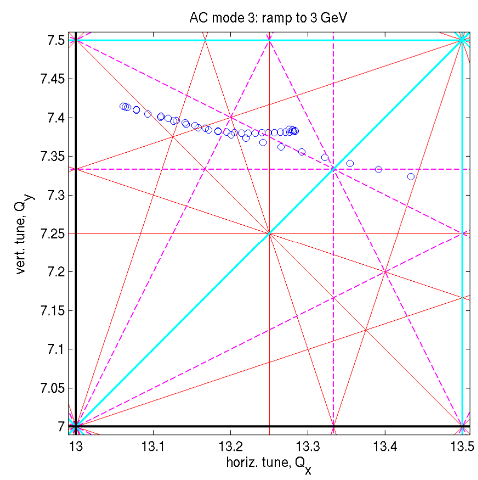
If there are errors and perturbations which are sampled periodically their effect can build up and **destroy the stability of motion**

The resonant condition defines a set of lines in the tune diagram

The **working point** has to be chosen away from the resonance lines, especially the lowest order one



5-th order resonance phase space plot
(machine with no errors)



Phenomenology of nonlinear motion (II)

Stable and unstable fixed points appears which are connected by separatrices

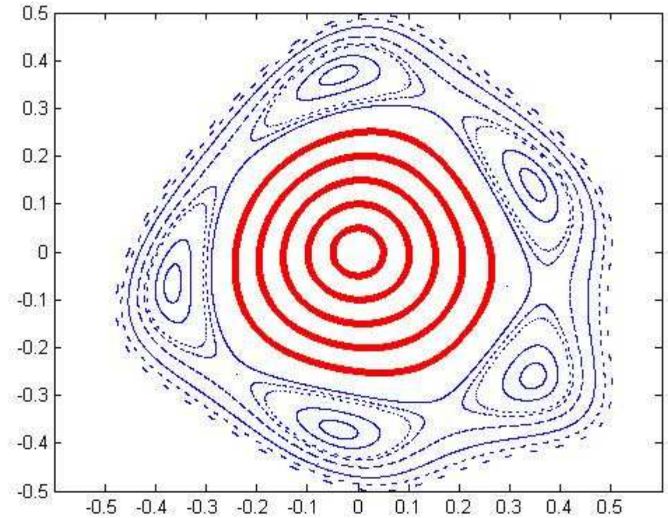
Islands enclose the stable fixed points

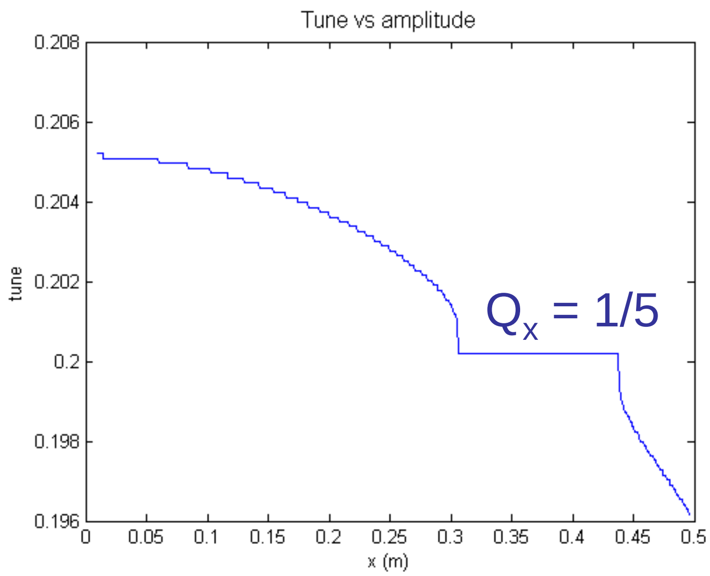
On a resonance the particle jumps from one island to the next and the tune is locked at the resonance value

regions of chaotic motion appear

The region of stable motion, called **dynamic aperture**, is limited by the appearance of unstable fixed points and trajectories with fast escape to infinity

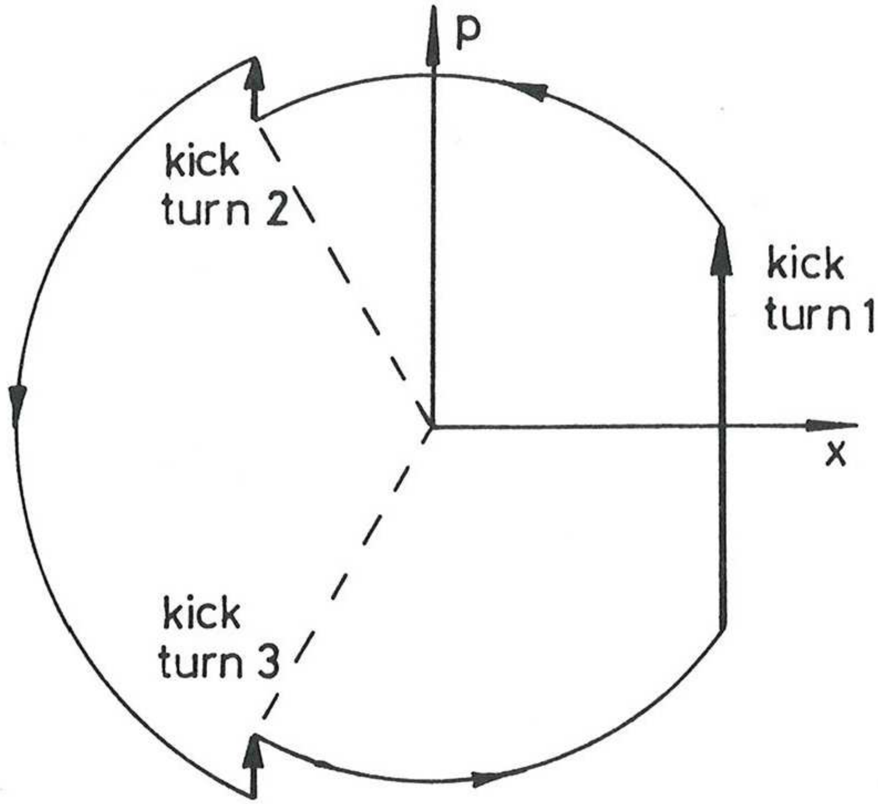
Phase space plots of close to a 5th order resonance





Simplified treatment of resonances

A simplified treatment of the resonance can be obtained by considering a single nonlinear element along the ring and looking at its effect on the charged particle motion in phase space:



The rest of the ring has no nonlinear element: the motion is just a rotation described by the unperturbed betatron tune Q, i.e.

$$x = A\cos(\varphi) \qquad \varphi = Q\theta$$

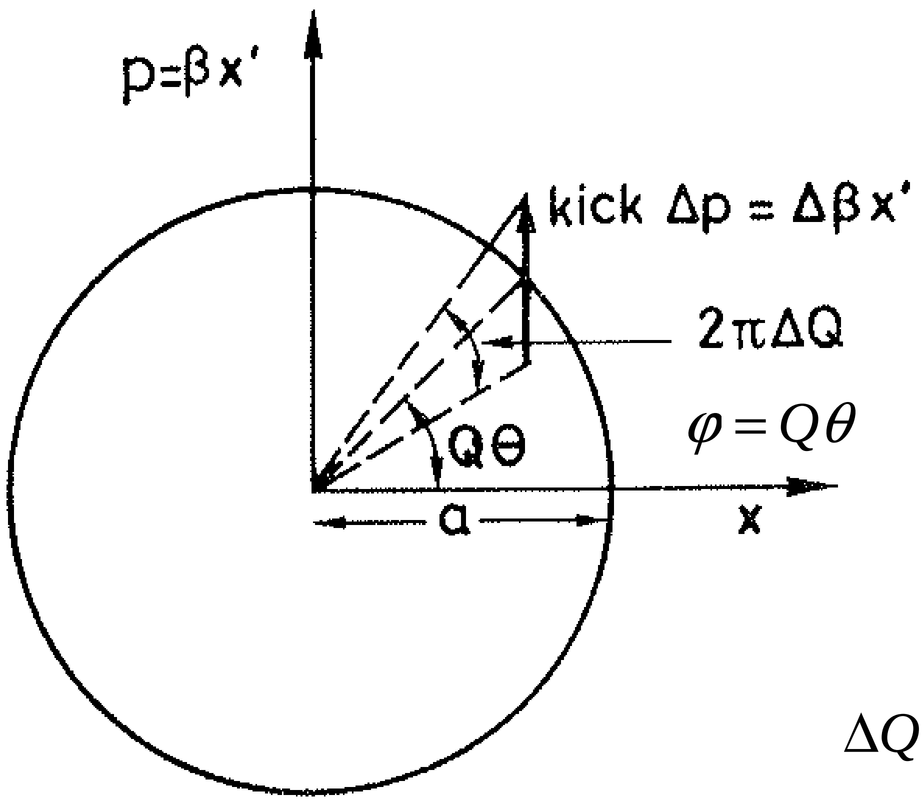
and θ (0 < θ < 2π) is the azimuthal along the ring.

When the particle reaches the nonlinear element it receives a kick proportional to the multipolar field error found

$$\Delta p = \frac{\beta L}{B\rho\, n!} \frac{\partial^n B_y}{\partial x^n} x^n$$

Example: second order resonance (I)

The effect of the kick can be computed analytically. Assume a quadrupole kick



$$\Delta p = \beta \Delta x' = \frac{\beta L \Delta g}{B\rho} x$$

The kick perturbs the amplitude and the phase

$$\Delta a = \Delta p \sin(\varphi)$$

$$\varphi = Q\theta$$

$$2\pi \Delta Q = \frac{\Delta p}{a} \cos(\varphi)$$

Substituting we obtain

$$\Delta Q = \frac{\beta L \Delta g}{2\pi B\rho} \cos^2(\varphi) = \frac{\beta L \Delta g}{4\pi B\rho} \left[\cos(2\varphi) + 1\right]$$

Over one turn the perturbed phase advance is $\Delta\varphi = 2\pi(Q + \Delta Q)$

Example: second order resonance (II)

The tune shift due to the kick $\Delta Q = \frac{\beta L \Delta g}{4\pi B\rho}\left[\cos(2\varphi)+1\right]$ has a constant term

and a term depending on the phase with which the charged particle meets the perturbing element.

Correspondingly, the perturbed tune $Q + \Delta Q$ changes at each turn, oscillating around the mean value with

$$\Delta Q = \frac{\beta L \Delta g}{4\pi B\rho}\cos(2\varphi)$$

with an amplitude

$$\delta Q = \frac{\beta L \Delta g}{4\pi B\rho}$$

If this band contains the half integer resonance, eventually, on a particular turn, the perturbed tune reaches the half integer resonance

$$Q + \Delta Q = \frac{p}{2}$$

Resonance stopband

When this happens the particle locks to the resonance since, in the subsequent turns, the perturbation to the tune will remain the same and will keep the perturbed tune fixed to the resonant value

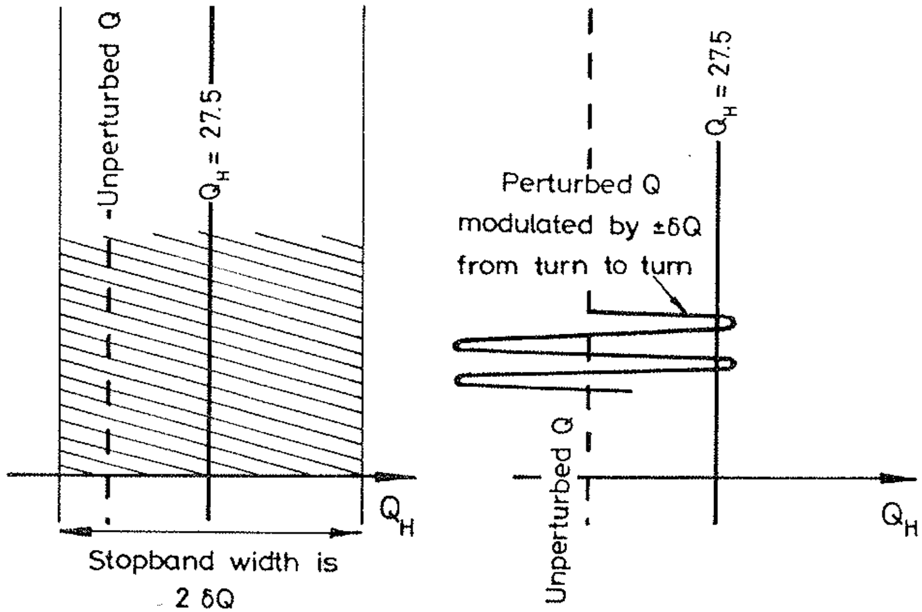
$$\Delta Q = \frac{\beta L \Delta g}{4\pi B\rho}\cos(2\varphi + 2\pi\, p) = \frac{\beta L \Delta g}{4\pi B\rho}\cos(2\varphi)$$

We can say that the half integer line has a width

$$\delta Q = \frac{\beta L \Delta g}{4\pi B\rho}$$

called resonance stopband.

All particles with tune within the stop band, will end up locked to the resonance



Once the particle is locked to the resonance the trajectory becomes periodic. This situation can lead to particle losses due to the second order resonance

Example: third order resonance

The kick due to a normal sextupole, can be written as $\Delta p = \beta \Delta x' = \frac{\beta L B''}{2B\rho} x^2$

Repeating the same procedure we can compute the tune shift due to the sextupole kick as

$$\Delta Q = \frac{\beta L B'' a}{16\pi B\rho} [\cos(3\varphi) + 3\cos(\varphi)]$$

If the tune is close to a third order resonance (Q = 1/3), within the stopband given by

$$\Delta Q = \frac{\beta L B'' a}{16\pi B\rho}$$

after a sufficient number of turns the tune will lock at the third order resonance, every three turn the motion will repeat identical and the amplitude will grow indefinitely.

Similarly it can be shown that an octupole excites a fourth order resonance, and a 2n-pole excites a n-th order resonance

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Hamiltonian of a charged particle in an accelerator

In a lattice made from dipoles, quads and sextupoles the Hamiltonian reads:

$$H(s) = \overbrace{\frac{p_x^2 + p_y^2}{2(1+\delta)}}^{\text{kinetic}} \overbrace{- b_1 x\delta + \frac{b_1^2}{2} x^2}^{\text{dipoles}} + \underbrace{\overbrace{\frac{b_2}{2}(x^2 - y^2)}^{\text{quads}}}_{H_2(s)} + \underbrace{\overbrace{\frac{b_3}{3}(x^3 - xy^2)}^{\text{sextupoles}}}_{H_3(s)} + \ldots$$

The goal is finding a sextupole distribution such that $\int_{cell} H_2 + H_3 \, ds$ becomes achromatic and linear:

- Independent of δ (chromaticity corrected)
- Linear and uncoupled ($\sim x^2, y^2$): cancellation of nonlinear kicks

Hamiltonian (II)

Introducing in $H(s)$ linear betaron oscillations for a flat lattice ($D_x = D$, $D_y = 0$):

$$x(s) = \sqrt{2J_x \beta_x(s)} \cos\phi_x(s) + D(s)\delta \qquad y(s) = \sqrt{2J_y \beta_y(s)} \cos\phi_y(s)$$

the powers of the trigonometric functions are turned into linear functions and the terms with the same argument (the modes) collected:

$$\int_{cell} [H_2(s) + H_3(s)] ds = \sum_{\substack{j=0 \\ j+k=m_x}}^{m_x} \sum_{\substack{l=0 \\ l+m=m_y}}^{m_y} \sum_{p=0}^{m_x} h_{jklmp}$$

with:

$$h_{jklmp} \propto \left[\sum_n^{N_{sext}} (b_3 L)_n \beta_{xn}^{\frac{j+k}{2}} \beta_{yn}^{\frac{l+m}{2}} D_n^p e^{i\{(j-k)\phi_{xn} + (l-m)\phi_{yn}\}} \right] - \left[\sum_n^{N_{quad}} (b_2 L)_n \beta_{xn}^{\frac{j+k}{2}} \beta_{yn}^{\frac{l+m}{2}} e^{i\{(j-k)\phi_{xn} + (l-m)\phi_{yn}\}} \right]_{p \neq 0}$$

The h_{jklmp} are called resonance driving terms since they generate angle dependent terms in the Hamiltonian that are responsible for the resonant motion of the particles. On the islands the betatron tune satisfy a resonant condition of the type

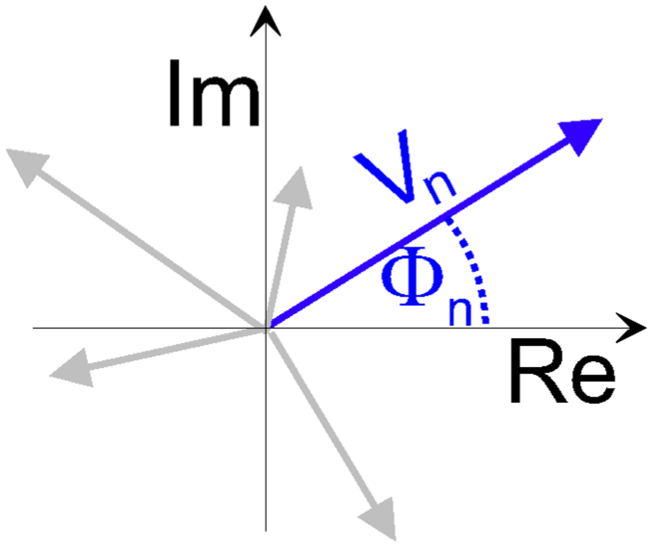
$a_x Q_x + a_y Q_y = r \rightarrow$ resonance (a_x, a_y) of order $a_x + a_y$ with $a_x = j - k$ and $a_y = l - m$

Hamiltonian modes

The Hamiltonian can be expressed abbreviating

$$\int [H_2(s) + H_3(s)]ds = \sum_n^{N_{sext}} V_n e^{i\Phi_n} [+\ldots\text{quads for } p \neq 0\ldots]$$

The Hamiltonian modes are sums of complex vectors $V_n, e^{i\Phi n}$ where each vector corresponds to a sextupole.



Sextupole$_n$ $\leftrightarrow$ complex vector $V_n, e^{i\Phi n}$:

Length $V_n = V_n(b_3, L, \beta_x, \beta_y, D)$

Angle $\Phi_n = \Phi_n(\varphi_x, \varphi_y)$

- $\Phi n = 0 \rightarrow$ tune shift
- $\Phi n \neq 0 \rightarrow$ resonances

We have to find suitable distribution of nonlinear magnetic elements along the ring, i.e. suitable functions $V_n(s)$ that reduce or cancel those driving terms which are stronger in the uncorrected machine.

First order Hamiltonian with sextupoles (I)

→ 2 phase independent terms → chromaticities:

$$h_{11001} = +J_x \delta \left[\sum_n^{Nsext} (2b_3L)_n \beta_{xn} D_n - \sum_n^{Nquad} (b_2L)_n \beta_{xn} \right] \to \xi_x$$

$$h_{00111} = +J_y \delta \left[\sum_n^{Nsext} (2b_3L)_n \beta_{yn} D_n - \sum_n^{Nquad} (b_2L)_n \beta_{yn} \right] \to \xi_y$$

→ 7 phase dependent terms → resonances: long term behaviour (many cells and many turns) $h^N = h$ for N cells, many cells and many turns $N \to \infty$:

$$|h^{\infty}_{jklmp}| = \frac{|h_{jklmp}|}{2\sin\pi(a_x Q_x^{cell} + a_y Q_y^{cell})}$$

$$a_x = (j-k) \quad a_y = (l-m)$$

$$h_{21000} = h^*_{12000} \to Q_x$$

$$h_{30000} = h^*_{03000} \to 3Q_x$$

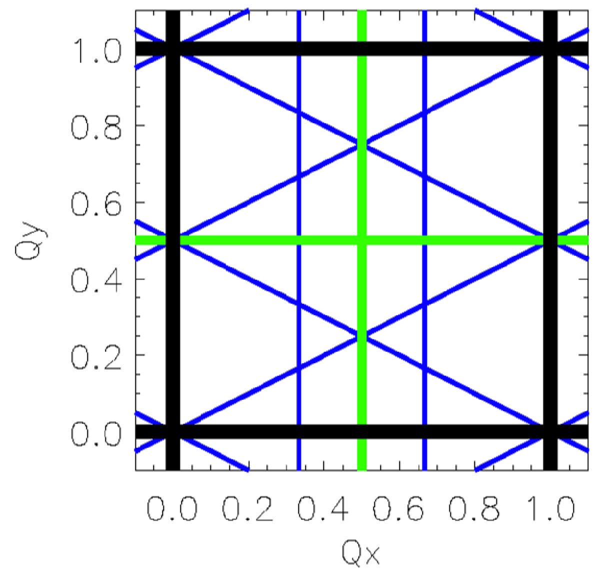
$$h_{10110} = h^*_{01110} \to Q_x$$

$$h_{10200} = h^*_{01020} \to Q_x + 2\,Qy$$

$$h_{10020} = h^*_{01200} \to Q_x - 2\,Qy$$

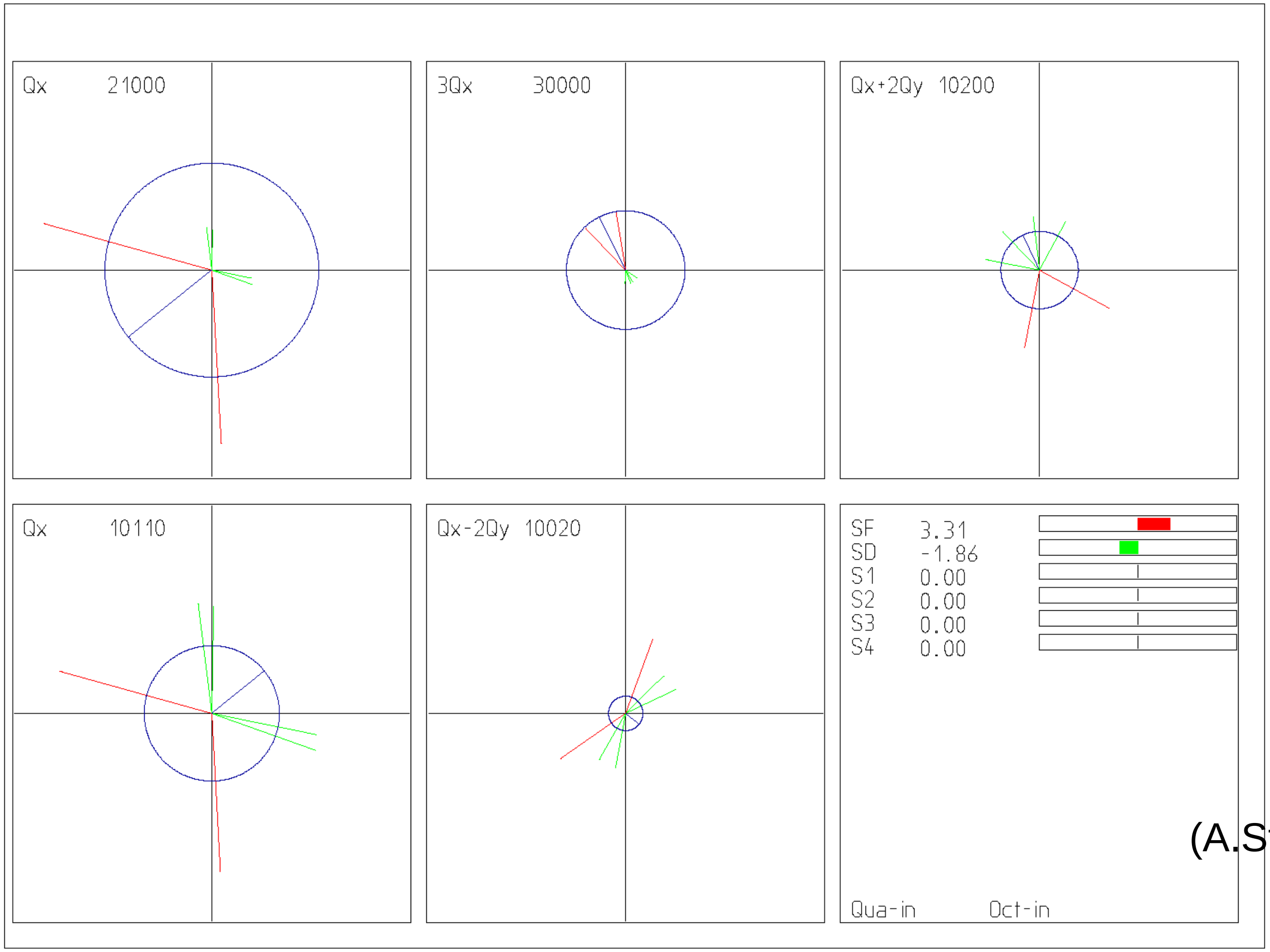
$$h_{20001} = h^*_{02001} \to 2Q_x$$

$$h_{00201} = h^*_{00021} \to 2Q_y$$



Example: sextupole optimization with OPA (I)

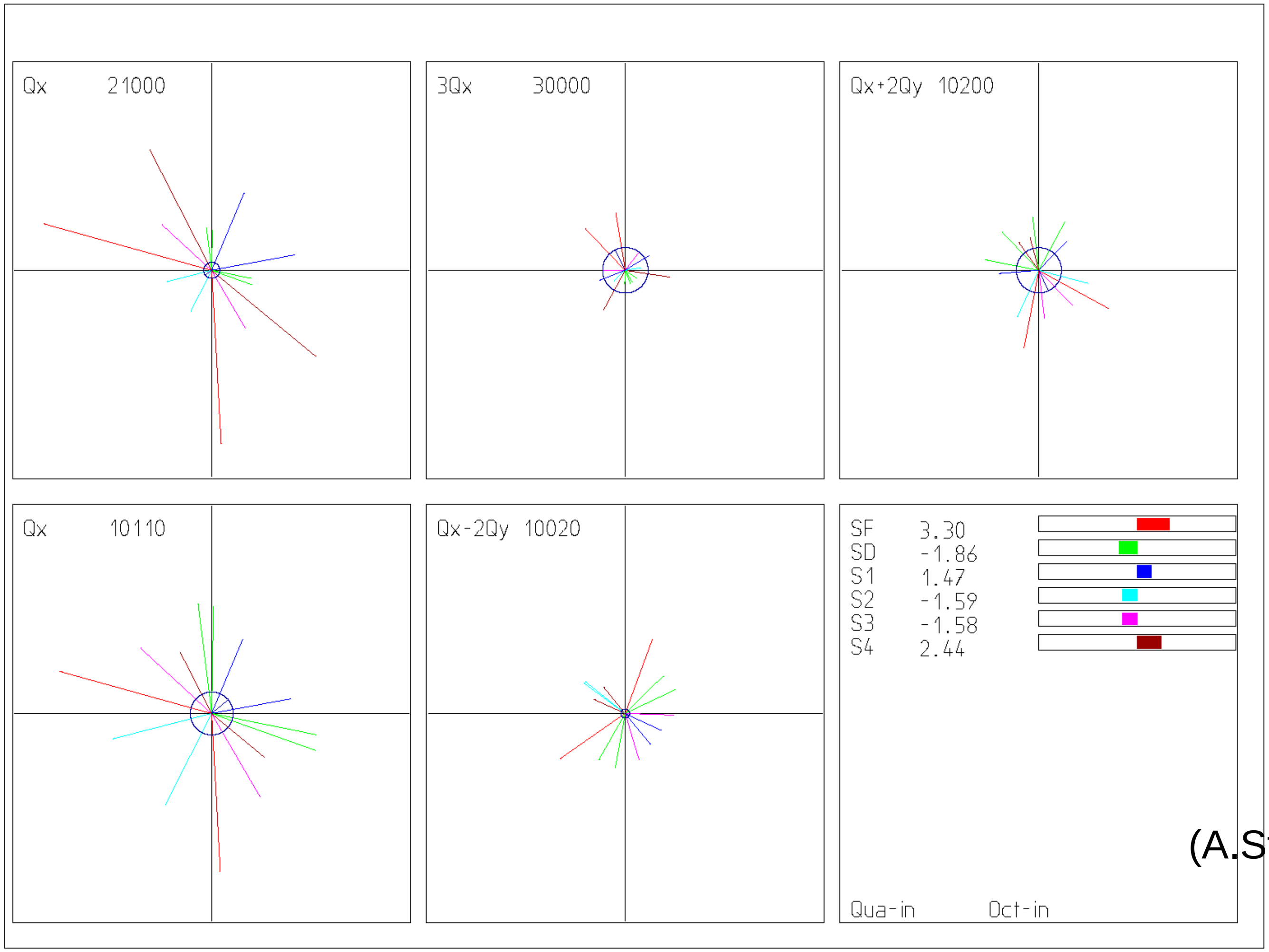
2 Sextupole families (ESRF standard cell)



(A.Streun, PSI)

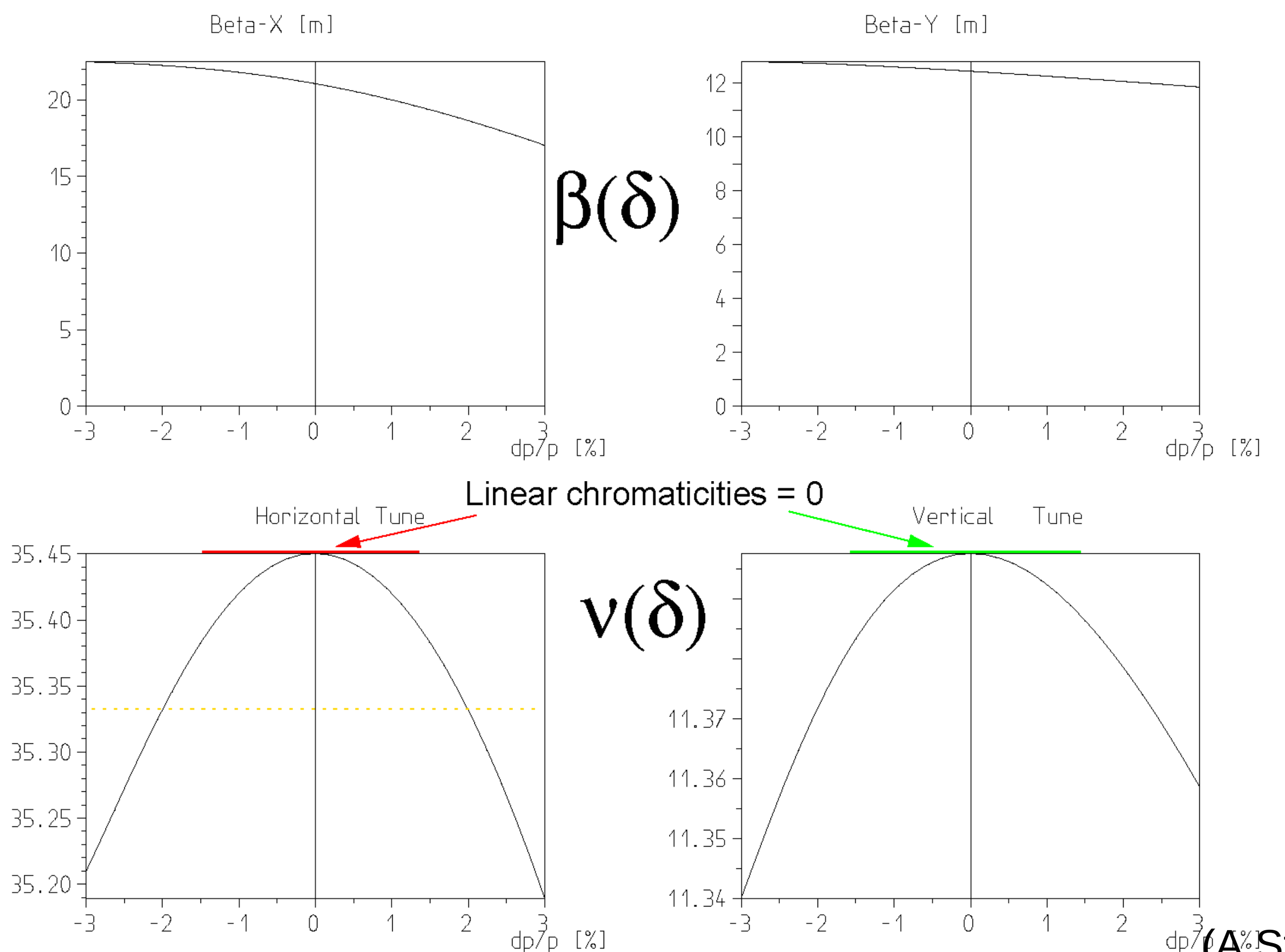
Sextupole optimization with OPA (II)

6 Sextupole families (4 *harmonic* families in straight sections)



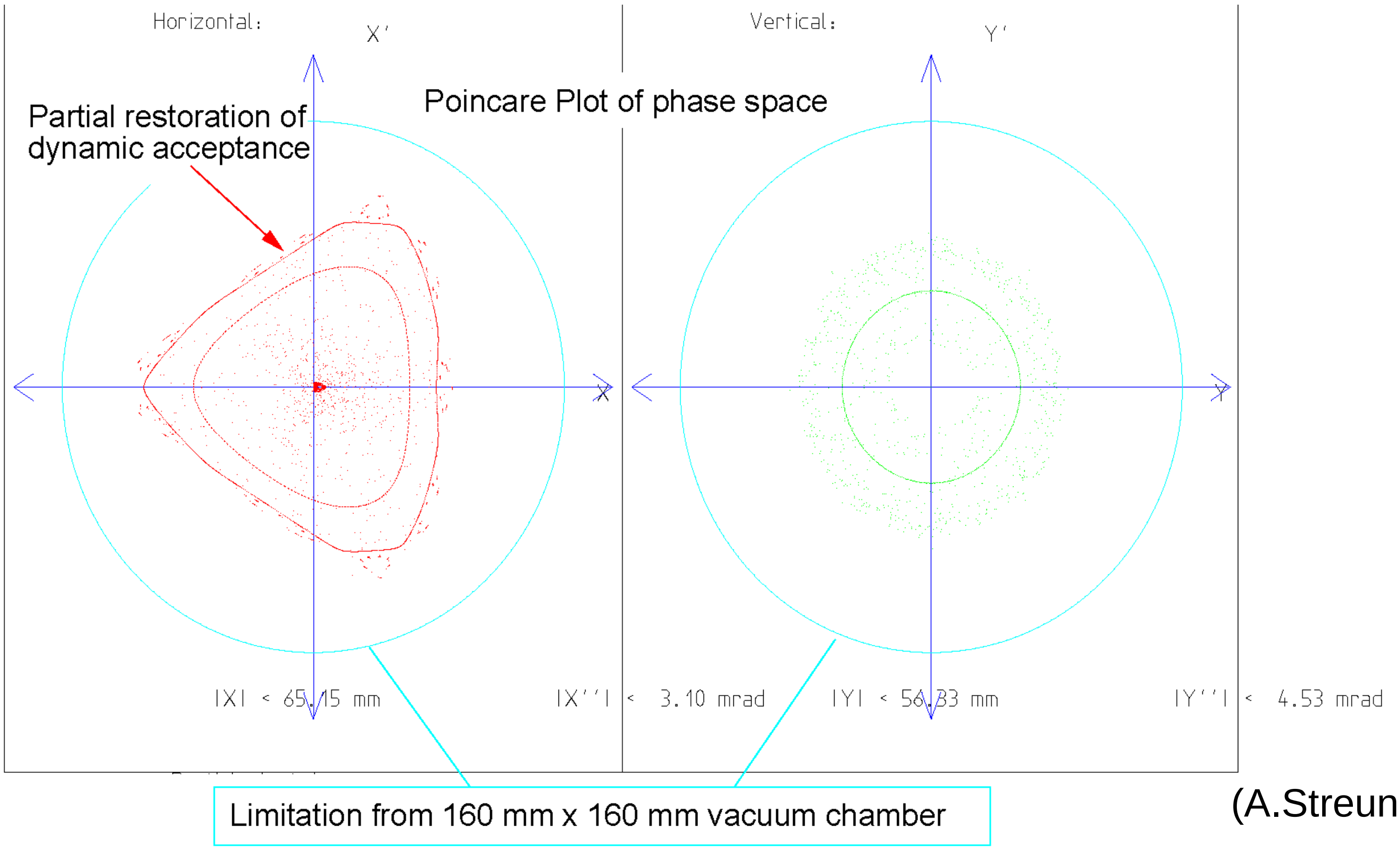
(A.Streun, PSI)

Sextupole optimization with OPA (III)



(A.Streun, PSI)

Sextupole optimization with OPA (IV)



(A.Streun, PSI)

Resonance compensation strategies

Systematic compensation of the resonance terms is done more efficiently choosing a suitable distribution of sextupole magnets along the ring with betatron phase relations that exploit the periodicity and symmetry of the machine.

e.g. typically two equal sextupoles at 60° phase adv ance apart compensate each other, in the $3Q_x$ resonance driving term.

However their effect on all the other resonances has to be assessed: the linear lattice design and the set-up of the nonlinear elements of a ring need an iterative work:

- decouple chromatic sextupoles → placement of 2 SF, SD families
- fix phase advances, exploit symmetry and periodicity →back to linear lattice design and machine layout
- place “harmonic” sextupoles → 4 to 8 families…
- minimize first order terms setting sextupole strenghts
- check by traking → dynamic aperture, frequency maps…
- iterate…

Tracking (I)

Most accelerator codes have tracking capabilities: MAD, MADX, Tracy-II, elegant, AT, BETA, Transport, …

Typically one defines a set of initial coordinates for a particle to be tracked for a given number of turns.

The tracking program launches the particle through the magnetic elements. Each magnetic element transforms the initial coordinates according to a given integration rule which depends on the program used, e.g. Transport (in MAD)

$$\vec{X} = (x, x', y, y', z, \delta) \ , \ \delta = \frac{\Delta P}{P_0}$$

$$\vec{X}_f = \mathbf{R}\vec{X}_i \qquad \text{Linear map}$$

$$x_{j,f} = \sum_k \mathrm{R}_{jk} x_{j,i} + \sum_{kl} \mathrm{T}_{jkl} x_{j,i} x_{l,i} + \sum_{klm} \mathrm{U}_{jklm} x_{j,i} x_{l,i} x_{m,i} + \cdots$$

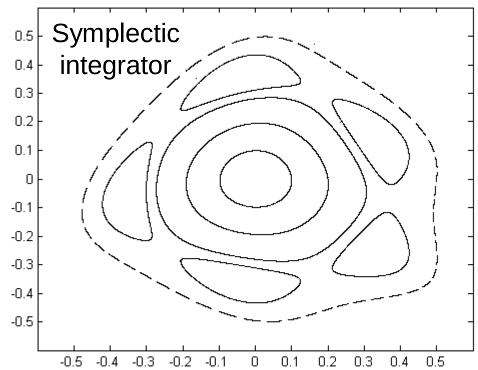
Nonlinear map up to third order as a truncated Taylor series

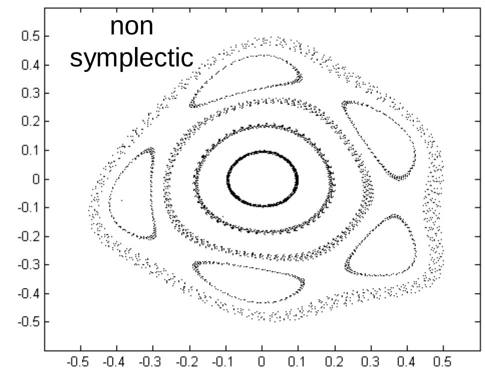
Tracking (II)

A Hamiltonian system is symplectic, i.e. the map which defines the evolution is symplectic (volumes of phase space are preserved by the symplectic map)

$\bar{x}_f = M(\bar{x}_i)$ M is symplectic transformation $J_{ab}(x_i) = \dfrac{\partial x_{a,f}}{\partial x_{b,i}}$ $J^T S J = S$

If the integrator is not symplectic one may found artificial damping or excitation effect





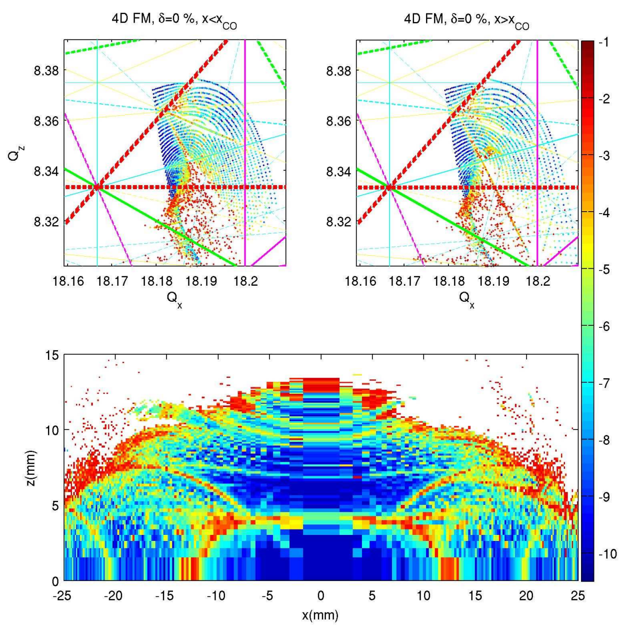
The well-known Runge-Kutta integrators are not symplectic. Likewise the truncated Taylor map is not symplectic. They are good for transfer line but they should not be used for circular machine in long term tracking analysis

Elements described by thin lens kicks and drifts are always symplectic: long elements are usually sliced in many sections.

Frequency Map Analysis

The Frequency Map Analysis is a technique introduced in Accelerator Physics form Celestial Mechanics (Laskar).

It allows the identification of dangerous non linear resonances during design and operation. Strongly excited resonances can destroy the Dynamic Aperture.



To each point in the (x, y) aperture there corresponds a point in the (Q_x, Q_y) plane

The colour code gives a measure of the stability of the particle (blu = stable; red = unstable)

The indicator for the stability is given by the variation of the betatron tune during the evolution: i.e. tracking N turns we compute the tune from the first N/2 and the second N/2

$$D = \log_{10}\sqrt{(Q_x^{(2)} - Q_x^{(1)})^2 + (Q_y^{(2)} - Q_y^{(1)})^2}$$

Bibliography

The slides of this class have been mostly taken from these two courses:

- A. Streun, Non-linearities in light sources, CAS 2003
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